Logo Symmetry Learning Task

Unit 5

Course

Mathematics I: Algebra, Geometry, Statistics

Overview

The **Logo Symmetry Learning Task** explores graph symmetry and odd and even functions. Students are asked to solve simple radical equations. In working with symmetry of graphs, students will apply the concepts of similarity and transformations of geometric figures inherent in the Grade 7 standards for geometry. The task offers students an in-depth discussion of even and odd symmetry of graphs of functions and transformations of graphs by reflection in the coordinate axes. These topics and the discussion of solving rational equations and quadratic equations of the form $x^2 - c = 0$, $c \geq 0$, reinforce topics from geometry, especially the topics of symmetry and transformation of geometric figures, similar triangles, and the Pythagorean Theorem.

Key Standards

MM1A1. Students will explore and interpret the characteristics of functions, using graphs, tables, and simple algebraic techniques.
c. Graph transformations of basic functions including vertical shifts, stretches, and shrinks, as well as reflections across the x- and y-axes.
d. Investigate and explain the characteristics of a function: domain, range, zeros, intercepts, intervals of increase and decrease, maximum and minimum values, and end behavior.
h. Determine graphically and algebraically whether a function has symmetry and whether it is even, odd, or neither.

MM1A3. Students will solve simple equations.
a. Solve quadratic equations in the form $ax^2 + bx + c = 0$ where $a = 1$ by using factorization and finding square roots where applicable.
d. Solve simple rational equations that result in linear equations or quadratic equations with leading coefficient of 1.

Possible Materials

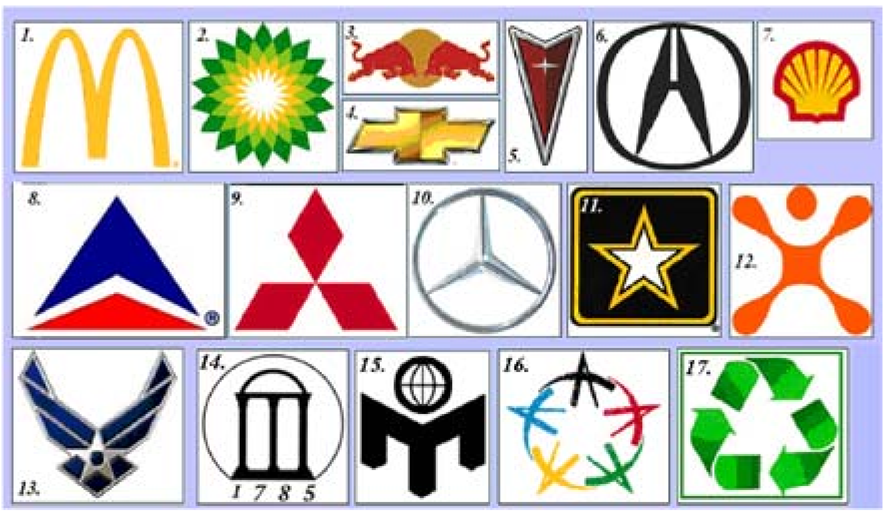
- Miras
- Patty paper
- Rulers/ Straightedges
- TI Navigator System
- Graphing calculators

Task

In middle school, students learned about line and rotational symmetry. Students can be asked to remember that a figure has line symmetry if there is a line that divides the figure into two parts that are mirror images of each other and that a figure has rotational symmetry if, when rotated by an angle of 180 degrees or less about its center, the figure aligns with itself.
A figure can also have point symmetry. A figure is symmetric about a single point if when rotated about that point 180 degrees it aligns with itself. So, rotational symmetry of 180 degrees is also symmetry about the center.

Sample Questions/Solutions

1.We all see many company logos everyday. These logos often have symmetry. For each logo shown below, identify and explain any symmetries you see.



Solution:
Except for the Logo 4, Logo 14 and Logo 16, each logo above has line symmetry with respect to the vertical that cuts the figure in half. If we omit the year "1785", the remaining figure in Logo 14 has symmetry with respect to the vertical that divides the figure in half. Logo 4 has rotational symmetry of 180°; this symmetry is also described as point symmetry about the center of the design.

Logos 9, 10, and 17 have rotational symmetry of 120° in the both the clockwise and counterclockwise directions. Logos 9 and 10 also have line symmetry about the lines obtained by rotating the vertical line through the center 120° clockwise and counterclockwise. Logo 11 has rotational symmetry of 72° and 144° in both the clockwise and counterclockwise directions. If we ignore the colors, Logo 16 has rotational symmetry of 72° and 144° in both the clockwise and counterclockwise directions. Logo 2 has rotational symmetry of 20°, 40°, 60°, 80°, 100°, 120°, 140°, 160°, and 180° in both the clockwise and counterclockwise directions. The logo also has line symmetry across any line obtained by rotating the vertical line through the center through a multiple of 10°. These lines are the diagonal lines that go from one of the outer points through the center to another outer point and the lines half way between any two of these diagonals.

2.

A textile company called “Uniform Universe” has been hired to manufacture some military uniforms. To complete the order, they need embroidered patches with the military insignia for a sergeant in the United States Army. To save on costs, Uniform Universe subcontracted a portion of their work to a foreign company. The machines that embroider the insignia design require a mathematical description. The foreign company incorrectly used the design at the right, which is the insignia for a British Sergeant. They sent the following description for the portion of the design to be embroidered in black.

Black embroidery instructions:

Vertical line	Restriction
$x = -7$	$0 \le y \le 6$
$x = 7$	$0 \le y \le 6$

Function	Domain
$y = \lvert x \rvert - 1$	$-7 \le x \le 7$
$y = \lvert x \rvert - 3$	$-7 \le x \le 7$
$y = \lvert x \rvert - 5$	$-7 \le x \le 7$
$y = \lvert x \rvert - 7$	$-7 \le x \le 7$

a. Match the lines in the design to the functions indicated in the table.

Comments:
Students may use graphing calculators or other technology to reproduce the design. In order to restrict the domain in a graph drawn by technology, divide the expression for the function by the Boolean expression $(-7 \le x)\cdot(x \le 7)$. When the Boolean expression is false, we have division by 0 and no graph is drawn. When the Boolean expression is true, the graph is drawn.

Solution:
The graphs match the functions listed in the table in order from top to bottom. The absolute value graph is shifted down 1 unit, 3 units, 5 units, and 7 units respectively.

b. When the design is stitched on a machine, wide black stitching is centered along the lines given by the equations above. The British Sergeant’s insignia has a light-colored embroidery between the lines of black. Write a description for the lines on which the light-colored stitching will be centered. Use a table format similar to that shown above.

Function	Domain
$y = \lvert x \rvert - 2$	$-7 \le x \le 7$
$y = \lvert x \rvert - 4$	$-7 \le x \le 7$
$y = \lvert x \rvert - 6$	$-7 \le x \le 7$

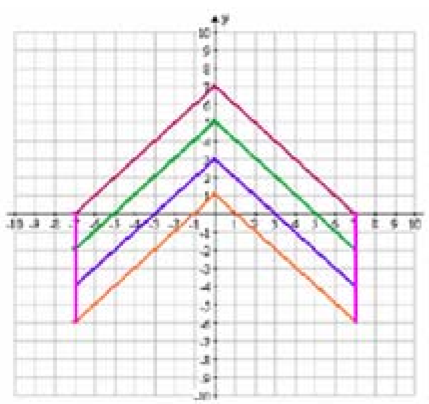
3.

Jessica, a manager at Uniform Universe, immediately noticed the design error when she saw some of the prototype uniforms. The sergeant's insignia was upside down from the correct insignia for a U.S. sergeant, which is shown at the right. Jessica checked the description that had been sent by the foreign contractor. She immediately realized how to fix the insignia. So, she emailed the foreign supplier to point out the mistake and to inform the company that the error could be corrected the by reflecting each of the functions in the x-axis.

Ankit, an employee at the foreign textile company, e-mailed Jessica back and included the graph below to verify that Uniform Universe would be satisfied with the new formulas.

a. What type of symmetry does the incorrect insignia have?

Line symmetry



b. If it is symmetric about a point, line, or lines, write the associated coordinates of the point or equation(s) for the lines of symmetry.

The line of symmetry is the y-axis, which is the vertical line with equaiton x = 0.

c. Does the corrected insignia have the same symmetry?
The correct insignia is symmetric with respect to the y-axis, that is, the vertical line x = 0.

d. Write the mathematical description of the design for the U.S. sergeant insignia, as shown in the graph below. Verify that your mathematical description yields the graphs shown.

Function	Domain
$y = -\lvert x \rvert + 7$	$-7 \le x \le 7$
$y = -\lvert x \rvert + 5$	$-7 \le x \le 7$
$y = -\lvert x \rvert + 3$	$-7 \le x \le 7$
$y = -\lvert x \rvert + 1$	$-7 \le x \le 7$

The correct insignia is described by reflecting each of the equations through the x-axis. To reflect a graph through the x-axis, we set the y-coordinate equal to the negative of the previous y-coordinate. So, to reflect y = |x| – 7 through the x-axis, we form the equation y = –(|x| – 7) = – |x| + 7.

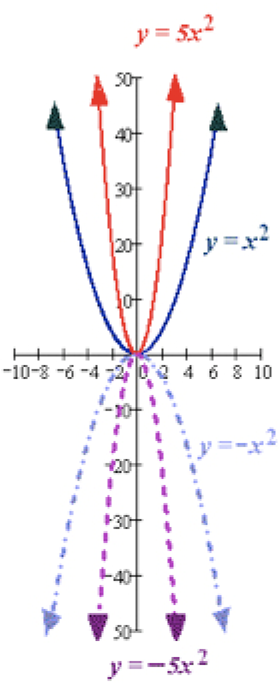
We can see that the graphs are the graph of y = – |x| shifted up 1 unit, 3 units, 5 units, and 7 units, respectively.

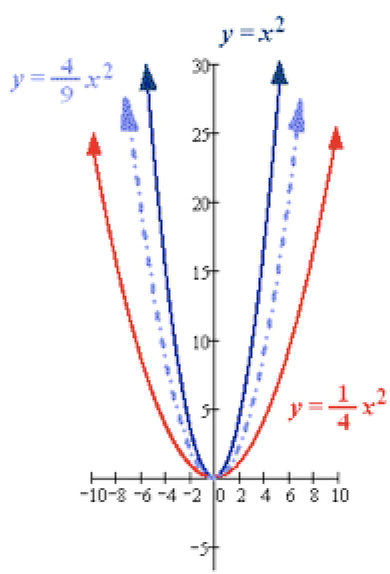
e. Let f denote any one of the functions graphed in the British sergeant's insigniz or the U.S. sergeant's insignia. Compare f(1) and f(–1), f(2.5) and f(–2.5), f(3.7) and f(–3.7). If x is a number such that 0 ≤ x ≤ 7, how do f(x) and f(– x) compare?

On each of the graphs, f(1) = f(–1), f(2.5) = f(–2.5), f(3.7) = f(–3.7). If x is a number such that 0 ≤ x ≤ 7, then f(x) = f(– x).

f. Let a be a constant other than the number 0 and let g denote the function whose formula is given by g(x) = ax^2. You studied the shapes of these graphs in Unit 1. Look back at some examples for particular choices of a. What type of symmetry do these graphs have? If x is a positive number, how do g(x) and g(– x) compare?

Graphs of some of the functions of the form g(x) = ax^2 considered in Unit 1, Painted Cubes Learning Task, are shown below. Each of the graphs has line symmetry with respect to the y-axis. For any positive number x, the value of g(x) is the same as the value of g(– x).





4.
We call a function f an even function if, for any number x in the domain of f, – x is also in the domain and f(– x) = f(x).

a. Suppose f is an even function and the point (3, 5) is on the graph of f. What other point do you know must be on the graph of f? Explain.

Comments:
Since (3, 5) is on the graph, 3 is in the domain and f(3) = 5. Then, by the definition of an even function, – 3 is also in the domain and f(– 3) = f(3) = 5 .

Solution:
(– 3, 5) is also on the graph of f

b. Suppose f is an even function and the point (– 2, 4) is on the graph of f. What other point do you know must be on the graph of f? Explain.

Comments:
Since (– 2, 4) is on the graph, – 2 is in the domain and f(– 2) = 4. Then, by the definition of an even function, – (– 2) = 2 is also in the domain and f(2) = f(– 2) = 4 .

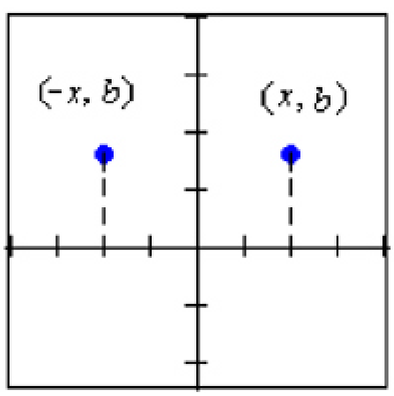
Solution:
(2, 4) is also on the graph of f

c. If (a, b) is a point on the graph of an even function f, what other point is also on the graph of f?

Comments:
Since (a, b) is on the graph, a is in the domain and f(a) = b. Then, by the definition of an even function, – a is also in the domain and f(– a) = – f(a) = – b.

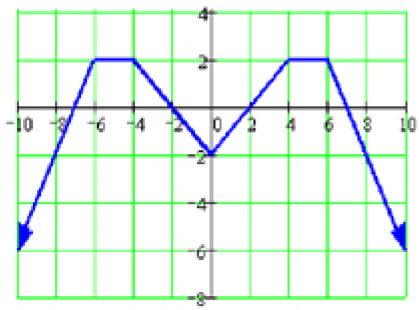
Solution:
(– a, b)

d. What symmetry does the graph of an even function have? Explain why.



Solution:
The graph of an even function has line symmetry with respect to the y-axis. For any number b, the points (x, b) and (– x, b) are at the same height on the grid and are equidistant from the y-axis; thus, they represent line symmetry with respect to the y-axis. Since f(x) and f(– x) are the same number, the points (x, f(x)) and (– x, f(– x)) are of the form (x, b) and (– x, b), and thus, are symmetric with respect to the y-axis. See the diagram above.

e. Consider the function k, which is an even function. Part of the graph of k is shown at the right. Using the information that k is an even function, complete the graph for the rest of the domain.



The graph of the function *k*

Comments: Since the graph of an even function is symmetric with respect to the y-axis, we reflect the part of the graph shown in the y-axis to obtain the rest of the graph, as shown.

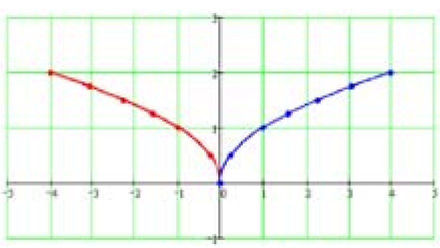
5.
When Jessica's supervisor, Malcom, saw the revised design, he told Jessica that he did not think that the U.S. Army would be satisfied. He pointed out that, while the revision did turn the design right-side up, it did not account for the slight curve in the lines in the real U.S. sergeant's insignia. He suggested that a square root function might be a better choice than an absolute value function and told Jessica to work with the foreign contractor to get a more accurate design. Jessica emailed Ankit to let him know that the design needed to be revised again to have lines with a curve similar to the picture from the U.S. Army website, as shown above, and suggested that he try the square root function.

a. Ankit graphed the square root function, $y = \sqrt{x}$, and decided to limit the domain to $0 \le x \le 4$. Set up a grid using a scale of ½ -inch for each unit, and graph the square root function on this limited domain. For accuracy, plot points for the following domain values: : 0, 1/4 , 1, 25/16 , 9/4 , 49/16 , 4.

Comments: The graph, using the scale of ½ -inch for each unit, is shown in blue below, after part b.

b. Ankit saw that his graph of the square root function (on the domain $0 \le x \le 4$) looked like the curve that forms the lower right edge of the British sergeant's insignia. He knew that he could reflect the graph through the x-axis to turn the curve over for the U.S. sergeant's insignia, but first he needed to reflect the graph through the y-axis in order to get the full curve to form the lower edge of the British sergeant's insignia. He knew that he needed to input a negative number and then get the square root of the corresponding positive number, so he tried graphing the function $y = \sqrt{x}$ on the domain $-4 \le x \le 0$. Reproduce Ankit's graph using the scale of ½-inch for each unit and using the opposites of the domain values in part a as some accurate points on the graph: 0, 1/4 , – 1, -25/16, -9/4 , -49/16 ,– 4.

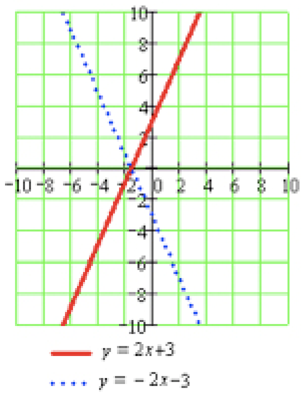
Comments: This graph is shown below in red on the same axes with the graph for part a.



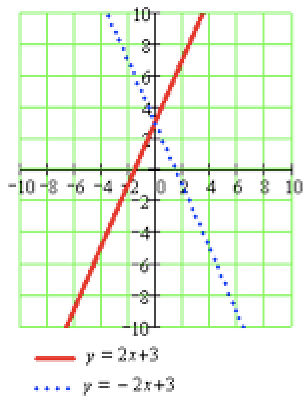
c. To see an additional example of reflecting a graph through the axes, graph each of the following and compare the graphs.

(i) $y = 2x + 3$ (ii) $y = -(2x + 3) = -2x - 3$ (iii) $y = 2(-x) + 3 = -2x + 3$

Reflection through the x-axis



Reflection through the y-axis



Let. $f(x) = 2x + 3$. Substituting $-f(x)$ for $f(x)$ gives a new function, $y = -(2x + 3)$, whose graph is obtained by reflecting the graph of $y = 2x + 3$ through the x-axis. Substituting $-x$ for x in the expression for f(x) gives a new function, $y = 2(-x) + 3$, whose graph is obtained by reflecting $y = 2x + 3$ Substituting $-x$ for x in the expression for f(x) gives a new function, $y = 2(-x) + 3$, whose graph is obtained by reflecting $y = 2x + 3$ through the y-axis.

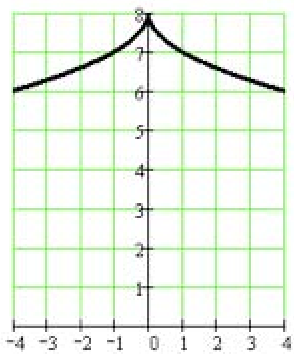
d. Given an equation of the form y = f(x), there is a specific process for creating a second equation whose graph is obtained by reflecting the graph of y = f(x) through the x-axis, and there is different process for creating an equation whose graph is obtained by reflecting the graph of of y = f(x) through the y-axis. Expalin these processes in your own words.

Comments: This item begins a process of helping students understand the similarities and differences in the two types of reflection. Reflecting the graph of a function through the x-axis is the simpler idea; it was introduced in Unit 1. Understanding reflection through the y-axis requires more sophisticated understanding of function notation and, hence, was delayed until Unit 5. One of the difficulties in helping students at this level understand the distinction is that, for three of the six basic functions studied in Mathematics I, the two types of reflection produce the same result. In particular, for the function f(x) = x
Given an equation of the form y = f(x).:
to reflect through the x-axis, form the equation y = -f(x)..

to reflect through the y-axis, form the equation y = f(-x)..

e. Write a formula for the function whose graph coincides with the graph obtained by reflecting the graph from part a in the x-axis and then shifting it up 8 units. Call this function f1, read “f sub one.”

Comments:
To reflect in the x-axis, we think of the equation y = $\sqrt{x}$ as y = f(x) with f(x) = $\sqrt{x}$. So, we form the equation y = – f(x), in particular, y = $-\sqrt{x}$.
Then we add 8 to shift the graph up 8 units. If students reverse the order so that they add 8 first and then take the negative of the formula, they will get the wrong answer with a y-intercept of -8 instead of +8. Students should be encouraged to understand that the order of the tranformations does matter.
Solution:
$f_1(x) = -\sqrt{x} + 8$



Graphs of f_1 and f_2

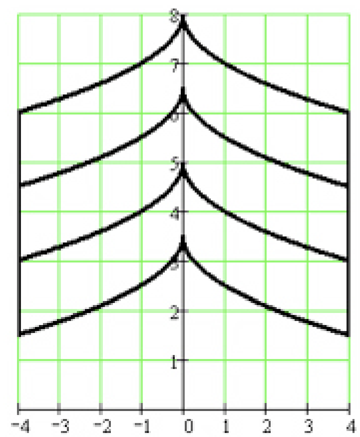
f. Write a formula for the function whose graph coincides with the graph obtained by reflecting the graph from part b in the x-axis and then shifting it up 8 units. Call this function f2, read “f sub two.” When graphed on the same axes, the graphs of the functions f1 and f2 give the curve shown above.

Comments:
This item revisits the idea introduced in part b of taking the square root of the opposite of x, denoted by – x. Students must understand (i) that the domain of the expression $\sqrt{-x}$ is the set of nonpositive numbers and (ii) that the graph of $y = -\sqrt{-x}$ is obtained by reflecting the graph of $y = \sqrt{-x}$ through the x-axis. With these two understandings, they will see that $-\sqrt{-x}$ cannot be simplified further and that it is not equal to $\sqrt{x}$.

Solution: $f_2(x) = -\sqrt{-x} + 8$

g. Ankit completed his mathematical definition for the U.S. Army sergeant's insignia and sent it to Jessica along with the graph shown at the right. Based on the graph and your work above, write Ankit's specifications for black embroidery of the insignia as shown in the graph.

Function	Domain
$y = -\sqrt{x} + 8$	$0 \le x \le 4$
$y = -\sqrt{-x} + 8$	$-4 \le x \le 0$
$y = -\sqrt{x} + 6.5$	$0 \le x \le 4$
$y = -\sqrt{-x} + 6.5$	$-4 \le x \le 0$
$y = -\sqrt{x} + 5$	$0 \le x \le 4$
$y = -\sqrt{-x} + 5$	$-4 \le x \le 0$
$y = -\sqrt{x} + 3.5$	$0 \le x \le 4$
$y = -\sqrt{-x} + 3.5$	$-4 \le x \le 0$



Sergeant insignia, U.S. Army

Vertical line	*Restriction*
$x = -4$	$1.5 \le y \le 6$
$x = 4$	$1.5 \le y \le 6$

Supporting Lessons Available :