Just Jogging

Unit 2

Course

Mathematics I: Algebra, Geometry, Statistics

Overview

In Unit 2, students develop skills in adding, subtracting, multiplying, and dividing elementary polynomial, rational, and radical expressions. By using algebraic expressions to represent quantities in context, students understand algebraic rules as general statements about operations on real numbers. The focus of the unit is the development of students' abilities to read and write the symbolically intensive language of algebra. Students will apply and extend all of the Grade 6 – 8 standards related to writing algebraic expressions, performing operations with algebraic expressions, and working with relationships between variable quantities. Students are assumed to have a deep understanding of linear relationships between variable quantities. Students should understand how to find the area of triangles, rectangles, squares, and trapezoids and know the Pythagorean theorem and need to have worked extensively with operations on integers, rational numbers, and square roots of nonnegative integers as indicated in the Grade 6 – 8 standards for Number and Operations.

By engaging in the tasks of this unit, students will experience opportunities to apply the basic function concepts of domain, range, rule of correspondence, and to interpret graphs of functions learned in Unit 1 of Mathematics I. The unit tasks begin with intensive work in writing linear and quadratic expressions to represent quantities in a real-world context. The initial focus is the development of students' abilities to read and write meaningful statements using the language of algebra. In the investigation of operations on polynomials, the special products of standard MM1A2 are studied as product formulas and interpreted extensively through area models of multiplication. *Factoring polynomials and solving quadratic equations by factoring are covered in a later unit, but the work with products and introduction of the zero factor property are designed to build a solid foundation for these later topics.*

The work with rational and radical expressions is grounded in work with real-world situations that show applications of working with such expressions. Students need to practice computational skills, but extensive applications are needed to demonstrate that the skills have important applicability in modeling and understanding the world around us. **The work with rational expressions includes this specific task's consideration of the application to calculating average speeds.**

Students are given many opportunities to use geometric reasoning to justify algebraic equivalence and to understand algebraic rules as statements about real number operations. Early in their study of algebra, students may have difficulty grasping the full content of abstract algebraic statements. Many of the questions in the tasks of this unit are intended to guide students to see how a geometric or other relationship from a physical context is represented by an algebraic expression. So, although some of the questions may seem very simple, it is important that they not be skipped. Gaining the ability to see all the information in an abstract algebraic statement takes time and lots of practice. Algebra is the language that allows us to make general statements about the behavior of the numbers and gaining facility with this language is essential for every educated citizen of the twenty-first century. Throughout this unit, it is important to:

• require students to explain how their algebraic expressions, formulas, and equations represent the geometric or other physical situation with which they are working.

• encourage students to come up with many different algebraic expressions for the same quantity, to use tables and graphs to verify expressions are equivalent, and to use algebraic properties to verify algebraic equivalence.

• make conjectures about relationships between operations on real numbers and then give geometric and algebraic explanations of why the relationship always holds or use a counterexample to show that the conjecture is false.

Key Standards

MM1A2. Students will simplify and operate with radical expressions, polynomials, and rational expressions.

Possible Materials

• calculators
• straightedges
• graph paper

Task

For distances of 12 miles or less, a certain jogger can maintain an average speed of 6 miles per hour while running on level ground.

Sample Questions/Solutions

1.

If this jogger runs around a level track at an average speed of 6 mph, how long in hours will the jogger take to run each of the following distances? [Express your answers as fractions of an hour in simplest form.]

(a) 3 miles (b) 9 miles (c) 1 mile (d) 1/2 mile (e) 1/10 mile

Comments:

This question has two purposes: engage students with fractions and have them calculate time given *the distance and the rate. Some students may know that* $t = \frac{d}{r}$ *but those who do not should be encouraged to use proportional reasoning. (For example, letting x represent the numerical part of the answer, we have for part (a)* $\frac{x\ hours}{1\ hour} = \frac{3\ miles}{6\ miles}$*.)*

The next question asks students to analyze their work in this question to see that time is calculated by dividing the distance by 6. Emphasize that students are answering in hours here. Some may want to answer in minutes for questions (c) – (e).

Solution:

(a) 1/2 hr
(b) 3/2 hr
(c) 1/6 hr
(d) 1/12 hr
(e) 1/60 hr

2.
Analyze your work in Question 1. Each answer can be found by using the number of miles, a single operation, and the number 6. What operation should be used? Write an algebraic expression for the time it takes in hours for this jogger to run x miles on level ground at an average speed of 6 miles per hour.

Comment:
If students need help, encourage them to try different operations with the number of miles and the number 6 and check to see which one works to give the answers they found in Question 1.

Solution: x/6

3.
Each day this jogger warms up with stretching exercises for 15 minutes, jogs for a while, and then cools down for 15 minutes. How long would this exercise routine take, in hours, if the jogger ran for 5 miles? [Express your answer as a fraction in simplest form.]

Comments:
Students may mix units and the leave the additional time for warm-up and cool down in minutes. Emphasize that the answer must be a single fraction that represents the number of hours.

Students may need reminding about the need for a common denominator when adding fractions of *hours.*

Solution: 4/3

4.
Let T represent the total time in hours it takes for this workout routine when the jogger runs for x miles. Write a formula for calculating T given x, where, as in Question 2, x is number of miles the jogger runs. Express the formula for T as a single algebraic fraction.

Comment:
Encourage students to look back to Question 3 and generalize the process they used to find the answer there.

Solution: $T = \frac{x+3}{6}$

5.
If the jogger skipped the warm-up and cool-down period and used this additional time to jog, how many more miles would be covered? Does this answer have any connection to the answer to question 4 above?

Comments:
Students should readily determine that an additional half hour of jogging will allow the jogger to run three more miles.

This question asks students to think back to the idea in Question 2 and think of the numerator of the expression in Question 4 as a number of miles. This connection should help them see that combining to a single fraction can give new insights.

Solution: 3

Suppose this same jogger decides to go to a local park and use one of the paths there for the workout routine one day each week. This path is a gently sloping one that winds its way to the top of a hill.

Comments:
This question and the next review skills with fractions but also preview the reasoning needed for Question 8.

Make sure that students understand that they will need to calculate uphill time and downhill time *separately since time is calculated by* $\frac{\text{distance}}{\text{rate}}$ *and the rates uphill and downhill are different.*

Students may be surprised that the total time is more than 1/3 hour (the time it would take the jogger to cover 2 miles on level ground) since averaging the two rates gives 6. Students may need to explore other back and forth travel with different rates to believe this; for example, they could calculate total travel time for a trip of 60 miles on the Interstate by going 30 miles and then returning, first at 60 miles per hour for the whole trip, and then by traveling at 75 miles per hour going and only 45 miles per hour coming back because of heavy traffic.

6.
If the jogger can run at an average speed of 5.5 miles per hour up the slope and 6.5 miles per hour going down the slope, how long, in hours, will it take for the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill? Give an exact answer expressed as a fraction in simplest terms and then give a decimal approximation correct to three decimal places.

Solution: $\frac{1}{5.5} + \frac{1}{6.5} = \frac{48}{143} \approx .336$

7.
If the jogger can run at an average speed of 5.3 miles per hour up the slope and 6.7 miles per hour going down the slope, how long, in hours, will it take for the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill? Give an exact answer expressed as a fraction in simplest terms and then give a decimal approximation correct to three decimal places.

Comment:
Finding the exact answer in simplest fractional form is not only good review of operations with fractions but it will also help student see that these answers agree with the answers that they get from their general algebraic fraction when they answer Question 8.

Solution: $\frac{1}{5.3} + \frac{1}{6.7} = \frac{1200}{3551} \approx .338$

8.
Write an algebraic expression for the total time, in hours, that it takes the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill if the jogger runs uphill at an

average speed that is c miles per hour slower than the level-ground speed of 6 miles per hour and runs downhill at an average speed that is c miles per hour faster than the level-ground speed of 6 miles per hour. Simplify your answer to a single algebraic fraction. Verify that your expression gives the correct answers for Questions 6 and 7.

Comment:
If students have wrong answers, have them attempt to verify that their formulas give the same answers for Question 6 and 7 so that they can see there is a problem and attempt to correct errors themselves.

Solution: $\frac{1}{6-c}+\frac{1}{6+c}=\frac{12}{36-c^2}$

Verification: for c = 0.5, $\frac{12}{35.75}=\frac{1200}{3575}=\frac{48}{143}$**; for c = 0.3,** $\frac{12}{35.51}=\frac{1200}{3551}$

9.
The average speed in miles per hour is defined to be the distance in miles divided by the time in hours spent covering the distance.

Comments:
This question addresses the mathematical issue of simple complex fractions that can be expressed as division of fractions.
This question also addresses the conceptual issue that may have already come up in Question 6, that the total time for the jogging routes of Questions 6 through 8 is more than a third of an hour, the time for 2 miles at 6 miles per hour. Using the simplified algebraic fraction from part (d), the discussion could begin by asking what happens if c = 0. It is important to help students see that the average rate is not calculated by averaging rates.
Also discuss giving meaning to negative values of c, that is, that the jogger would run faster uphill and slower downhill and consider what happens to the expression and the physical situation when c is 6 or –6. Students should see that the expression is undefined precisely when the story breaks down. They should also then realize that there are difficulties with meaning when c is greater than 6 or less than –6 (because one of the rates is negative) and, hence, see that a reasonable domain for c restricts it to the open interval (–6, 6).

In discussing part (e), it is appropriate to introduce the idea that the expression in (d) can be used to write a function A that gives average speed A(c) when the rate decreases by c miles per hour going uphill and increases by c miles per hour going downhill. Then the verification is a matter of calculating A(0.5) and A(0.7).
It is anticipated that students will use calculators for parts (b) and (c). In doing so, they originally use their decimal approximations from Questions 6 and 7, they will get slightly different answers that the answers from the expression in part (d). If so, discuss the issue of error introduced by rounding at an intermediate step in a calculation.

(a) What is the jogger's average speed for a two mile trip on level ground?

Solution: $\dfrac{2}{\frac{1}{3}} = 6$ **miles per hour**

(b) What is the jogger's average speed for the two mile trip in question 6?

Solution: $\dfrac{2}{\frac{48}{143}} = \dfrac{143}{24} = 5.958333\ldots$ **miles per hour**

(c) What is the jogger's average speed for the two mile trip in question 7?

Solution: $\dfrac{2}{\frac{1200}{3551}} = \dfrac{3551}{600} = 5.158333\ldots$

(d) Write an expression for the jogger's average speed over the two-mile trip (one mile up and one mile down) when the average speed uphill is c miles per hour slower than the level-ground speed of 6 miles per hour and the average speed downhill at an average speed that is c miles per hour faster than the level-ground speed of 6 miles per hour. Express your answer as a simplified algebraic fraction.

Solution: $\dfrac{2}{\frac{12}{36-c^2}} = \dfrac{36-c^2}{6}$

(e) Use the expression in part (d) to recalculate your answers for parts (b) and (c)? What value of c should you use in each part?

Solution:
Answers should be the same as previously found for parts (b) and (c).

10.

For what value of *c* would the jogger's average speed for the two-mile trip (one mile up and one mile down) be 4.5 miles per hour? For this value of *c*, what would be the jogger's average rate uphill and downhill?

Comments:
This question applies previous work with quadratic equations and previews the rational equations to come in a later unit.
Students should discuss whether this is a very realistic situation for a "gently sloping" jogging path.

Solution:

Equation: $\frac{36-c^2}{6}=4.5$**; solution: c = 3 or –3. For c = 3, the jogger's average speed would be 3 miles per hour uphill and 9 miles per hour downhill. For c = –3, the jogger's average speed would be 9 miles per hour uphill and 3 miles per hour downhill.**